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Optimization for Minimum Sensitivity to Uncertain Parameters

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A procedure to design a structure for minimum sensitivity to uncertainties in problem parameters is described. The approach is to directly minimize the sensitivity derivatives of the optimum design with respect to fixed design parameters using a nested optimization procedure. The procedure is demonstrated for the design of a bimetallic beam for minimum weight with insensitivity to uncertainties in structural properties. The beam is modeled with finite elements based on two-dimensional beam analysis. A sequential quadratic programming procedure used as the optimizer supplies the Lagrange multipliers that are used to calculate the optimum sensitivity derivatives. The procedure is validated by comparing the optimization results to parametric studies.

Introduction

PARAMETERS are characteristics of an optimization problem that are normally held constant during the optimization process. In practical design situations, the exact values of the parameters may not be known (e.g., loads, material properties, and manufacturing errors). Uncertainties in parameters may cause variation in the outcome of the design process; for example, uncertainties in the constraints may cause the design to be unacceptable. Unfortunately, in many cases the uncertainties are very difficult and costly to control. It would be advantageous to approach the design task using the key principle of quality design as formulated by Taguchi,[1] who proposed that designs be as insensitive as possible to variations in parameters that affect their behavior. In essence Taguchi advocated making designs insensitive to variations rather than controlling such variations.

Efforts have been directed toward developing design processes wherein the performance of the final product will be minimally affected by variations in parameters. Several formulations approach the problem by imposing safety margins on the constraints to avoid entering the infeasible region. For example, in Ref. 2 a worst-case approach is used wherein a specified tolerance on the uncertain design variables reduces the feasible design space. The objective is to stay as close as possible to the original optimum and to remain within the newly defined feasible region. A padding factor, proportional to the gradient of the constraint, is added to the constraints in Ref. 3. The constraints are updated throughout the optimization process using linear and nonlinear approximations. In Refs. 4 and 5 the authors proposed designing for feasibility robustness. Given specified tolerances on design variables and parameters, variations in the constraints are predicted using worst-case and statistical analysis. This maintains a certain probability of remaining feasible throughout the design process in spite of fluctuations in the variables and parameters.

These methods effectively reduce the size of the feasible region resulting in trading a less optimum value of the objective function for a higher degree of robustness. In Refs. 2, 4, and 5 this concept was enhanced by adding tolerance ranges on specified controllable variations as variables in the objective function so that the values of the tolerances were chosen during the design process. Reference 6 labeled this approach a tolerance allocation problem. The method was applied to the design of a simple two-bar truss to obtain a tradeoff between the tolerances on the structural dimensions and the manufacturing cost. As the tolerances decrease, the cost of manufacturing the product increases. In Ref. 7 a second-order tolerance model was incorporated into a nonlinear optimization procedure that improved the formulation by including a measure of function skewness in the design process.

Another approach to decreasing the effects of uncertainties in design is to directly minimize the sensitivity of the performance of a design with respect to the uncertainties. Reference 8 minimized the sensitivity derivative of the response of a system with respect to uncertain parameters using shape design variables. Reference 9 proposed that tradeoffs be made in the design objectives to ensure a degree of insensitivity to uncontrolled parameter variations. This approach was referred to as a design for latitude, and it was handled with a multiple objective function that included the sensitivity of the design with respect to the uncertain parameters. Reference 10 also used multiobjective optimization methods to minimize sensitivities to variations in design parameters by incorporating sensitivity derivatives of the objective and constraints into the objective function or by constraining them to be less than a specified value. Reference 11 applied goal programming methods to minimize multiple objectives, including the sensitivities of parameters and design variables to uncertainties. Reference 12 utilized fuzzy set theory to define a multiple objective design optimization problem dealing with uncertainty.

The distinctive feature of the approach described in this paper is that the design itself is desensitized to parameters that are uncertain or subject to change. This approach may minimize the need for redesign if the parameters vary from the value they have during the design process. The approach taken in this paper extends the concept of minimizing behavior sensitivity derivatives to that of minimizing optimum sensitivity derivatives (OSD) with respect to uncertain problem parameters using a nested optimization procedure. OSD are calculated analytically using the methods described in Refs. 13 and 14. The procedure is demonstrated for the design of a bimetallic beam for minimum weight and frequency constraints with insensitivity to uncertain structural properties. The beam is modeled with finite elements based on classical two-dimensional beam theory. A sequential quadratic programming procedure[15] used for the optimization supplied the Lagrange multipliers that were used to calculate the OSD using the method described in Refs. 13 and 14.

Approach: General Formulation

The overall procedure is based on two nested optimization loops depicted in Fig. 1. The primary optimization problem is solved in the inner loop (box 1) yielding a constrained optimum comprising optimum values of the objective function F and the design variables X. The term Y represents the set of auxiliary design variables that are used to control the sensitivity of the design. The parameters P that are held constant during the inner-loop design process may have uncertainty associated with their values. The inner-loop design is a function of these parameters and is, therefore, sensitive to variations or uncertainties in their values. In box 2 the sensitivities of the inner-loop design with respect to the uncertain parameters are calculated. These derivatives are represented by $\mathrm{d}F_{\mathrm{opt}}/\mathrm{d}P$, denoted $\dot{F}_{\mathrm{opt}}$ for simplicity. In box 3 the search algorithm determines the

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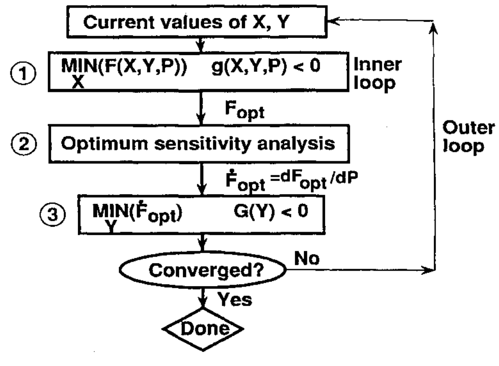


Fig. 1 Flowchart of overall procedure.

values of Y that minimize the sensitivity of the inner-loop design subject to the constraints $g(Y)$. In other words, all of the inner-loop optimum designs, the method seeks the one that has the lowest value for the optimum sensitivity derivatives. This part of the procedure is called the outer loop. Because there are two nested optimization loops, this optimization problem may be expensive to solve. The computational expense may be amplified since minimizing a sensitivity involves minimizing a function that already depends on the derivatives. Use of a gradient-driven search algorithm would imply working with computationally expensive second derivatives. Therefore, a nongradient-based search algorithm is used in the outer loop.

Example Problem

A simple but cogent example is used to demonstrate the procedure and its potential. A beam (Fig. 2) of length $L = 100.0$ in. is clamped at both ends. The beam is made up of two sections. From the left end to a distance L_1 from the left end, the beam has Young's modulus E_1, a density ρ_1, and height h_1. The remainder of the beam has E_2, ρ_2, and h_2. The sections have rectangular cross sections of width B_1 and B_2. The primary objective in the inner loop is to reduce the weight of this beam subject to frequency constraints. The design variables are the heights of the two beam sections, h_1 and h_2. L_1 is the auxiliary design variable used in the outer loop to control the sensitivity of the design. The uncertain parameters are the density of the first beam section ρ_1 and Young's modulus of the second beam section E_2. The nominal values of these material properties are given in Table 1. The flowchart in Fig. 3 shows the overall procedure with the details for this problem. The discussion of this figure is separated into three parts: the inner-loop formulation (box 1); the optimum sensitivity analysis (box 2); and the outer-loop formulation (box 3).

Inner-Loop Formulation

In box 1 the inner optimization problem minimizes the weight of the beam subject to frequency constraints using a sequential quadratic programming (SQP) algorithm. The objective function F is given as

$$F = \sum_{i=1}^{2} \rho_i L_i B_i h_i \qquad (1)$$

where ρ_i, B_i, L_i, and h_i are the density, width, length, and height of the ith section of the beam, respectively. In the inner-loop optimization, the design variables are h_1 and h_2, whereas L_1 is a parameter. Constraints are lower and upper bounds on the fundamental bending frequency ω. The constraints are represented by

$$1.0 - (\omega/\omega_l) \leq 0 \qquad (\omega/\omega_u) - 1.0 \leq 0 \qquad (2)$$

where ω_l and ω_u are the lower and upper bounds on the frequency, respectively.

Optimum Sensitivity Analysis

In box 2 the OSD are calculated from Eq. (3) using the quasi-analytical method of Refs. 13 and 14,

$$\frac{\mathrm{d}F_{\mathrm{opt}}}{\mathrm{d}P} = \frac{\partial \bar{F}}{\partial P} + \bar{\lambda}^T \frac{\partial \bar{g}_a}{\partial P} \qquad (3)$$

Table 1 Beam material properties

	Section 1	Section 2
Density, lb/in.^{3}	0.100	0.056
Young's modulus, psi	2.0E+7	1.0E+7
Width in.	4.0	4.0
	L = 100 in.	

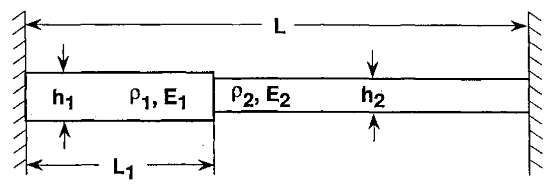


Fig. 2 Bimaterial beam problem, uncertain parameters, ρ_1, E_2.

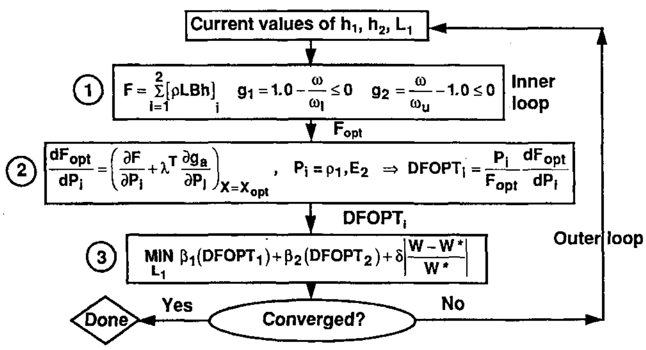


Fig. 3 Test problem formulation.

The overbars are used to denote quantities at the optimum, $\bar{g}_a$ represents the constraints that are active at the optimum, $\bar{\lambda}$ is the Lagrange multiplier, and P represents the parameter. The Lagrange multipliers are byproducts of the SQP procedure used to solve the inner-loop optimization problem. The success of this procedure is critically dependent on a highly converged inner-loop optimization resulting in accurate Lagrange multipliers.

Outer-Loop Formulation

In the outer loop denoted by box 3, a combination of optimum weight and sensitivity of the inner-loop optimum design is minimized with respect to the uncertainty in ρ_1 and E_2 using L_1 as the auxiliary design variable. In other words, of all of the minimum weight designs that satisfy the constraints, the procedure seeks the one that is least sensitive to uncertainties in ρ_1 and E_2. This can be done using a multiobjective function shown in Eq. (4). The first two terms are the magnitudes of the two optimum sensitivity derivatives having weighting factors β_1 and β_2. The third term represents the deviation from a reference weight W^* used for normalization purposes and δ is a weighting factor. Thus,

$$\mathrm{OBJ} = \beta_1(\mathrm{DFOPT}_1) + \beta_2(\mathrm{DFOPT}_2) + \delta|(W - W^*/W^*)| \qquad (4)$$

DFOPT_1 and DFOPT_2 are normalized OSD

$$\mathrm{DFOPT}_1 = \frac{\rho_1}{F_{\mathrm{opt}}} \frac{\mathrm{d}F_{\mathrm{opt}}}{\mathrm{d}\rho_1}$$
$$\mathrm{DFOPT}_2 = \frac{E_2}{F_{\mathrm{opt}}} \frac{\mathrm{d}F_{\mathrm{opt}}}{\mathrm{d}E_2} \qquad (5)$$

where $\mathrm{d}F_{\mathrm{opt}}/\mathrm{d}\rho_1$ and $\mathrm{d}F_{\mathrm{opt}}/\mathrm{d}E_2$ are given by Eq. (3). DFOPT_1 and DFOPT_2, sometimes referred to as logarithmic derivatives, are valuable in that they measure a percent change in the objective function due to a percent change in the design variables. The weighting factors β_1, β_2, and δ in Eq. (4) allow tradeoffs between the optimum objective function that is achieved in the inner loop and the level of sensitivity of this design to the uncertain parameters. The factor δ is used to prevent the procedure from decreasing the sensitivity of the design at the expense of a large weight increase. The values of the weighting factors reflect the relative importance of desensitizing the design vs reaching extreme values for the primary problem objective function. The values of δ and β that are problem dependent are chosen arbitrarily. The results will show that the outcome of this procedure is very dependent on the values of these weighting factors. A rational method for obtaining the value of W^* is to optimize the weight of the beam subject to frequency constraints using both the inner- and outer-loop design variables simultaneously.

Minimization of the outer-loop objective is performed using Powells' method, a nongradient-based search algorithm.[16] It was decided to use Powells' method because it is typically used for problems with small numbers of design variables as in the case of this example problem, however, any nongradient-based search algorithm may be used.

Analytical Model

The beam is modeled with 10 finite elements that are based on a classical two-dimensional beam theory. The degrees of freedom include lateral displacements and slopes. No torsional or extensional degrees of freedom were included, and transverse shear and rotatory inertia have been neglected. As the value of L_1 was varied, each section of the beam was divided into elements such that a grid point always existed at L_1. Each section had a minimum of two elements.

Results

Results of Optimization Procedure

To assess the validity of the nested optimization procedure, a study was conducted in which the parameter L_1 was varied in the outer loop and only the inner-loop optimization was performed. Figure 4 represents a series of discrete calculations of OBJ [Eq. (4)] vs L_1/L for $\beta_1 = \beta_2 = 10$, $W^* = 30$, and $\delta = 1$. Each data point represents a minimum weight optimization performed in the inner loop and an optimum sensitivity analysis calculation. The minimum indicated in the figure corresponds to $L_1/L = 0.36$ for which the beam has a minimum weight design and is least sensitive to uncertainties in the parameters ρ_1 and E_2.

The optimization procedure of Fig. 3 produced a value for L_1/L of 0.364. Values associated with the initial and final designs are given in Table 2. These results indicate that the sensitivity derivative of the optimum weight with respect to ρ_1 (DFOPT_1) could be decreased by nearly a factor of three. The cost of this dramatic decrease in sensitivity was a 40% increase in weight and a slight increase in the sensitivity with respect to E_2. In this case, β_1 and β_2 were a factor of 10 larger than δ. It may be advantageous to use unequal weighting factors for the OSD to emphasize minimization of one or the other or to increase the value of δ to emphasize minimization of the weight. This trend may be altered depending upon the choice of the weighting factors β_1, β_2, and δ. The following section will discuss the role of the weighting factors in tailoring the problem.

Table 2 Initial and final optimum weight designs for $\delta = 1$, $\beta_1 = \beta_2 = 10$, and $W^* = 30$

	L_1/L	OBJ	h_1, in.	h_2, in.	Weight, lbs	DFOPT_1	DFOPT_2
Initial	0.50	7.7	0.53	3.6	51.0	0.38	0.32
Final	0.36	7.1	0.28	4.7	71.0	0.13	0.44

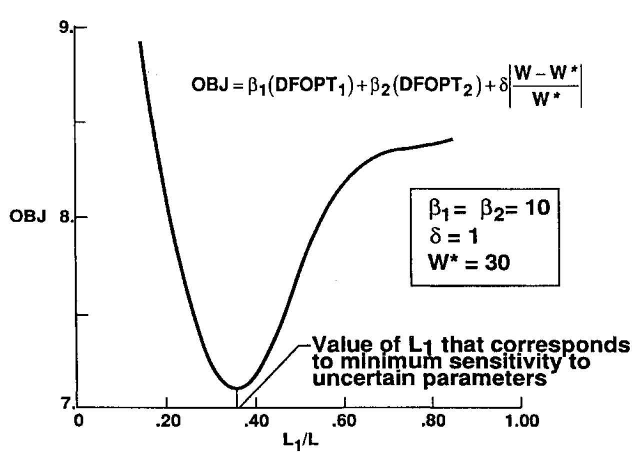
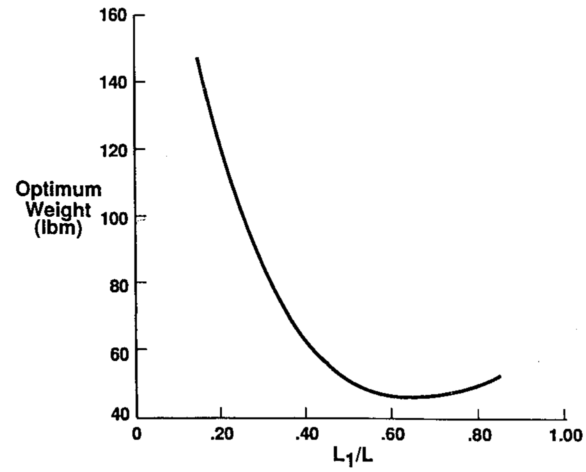


Fig. 4 Curve of discrete calculations of outer-loop objective function OBJ vs L_1/L.

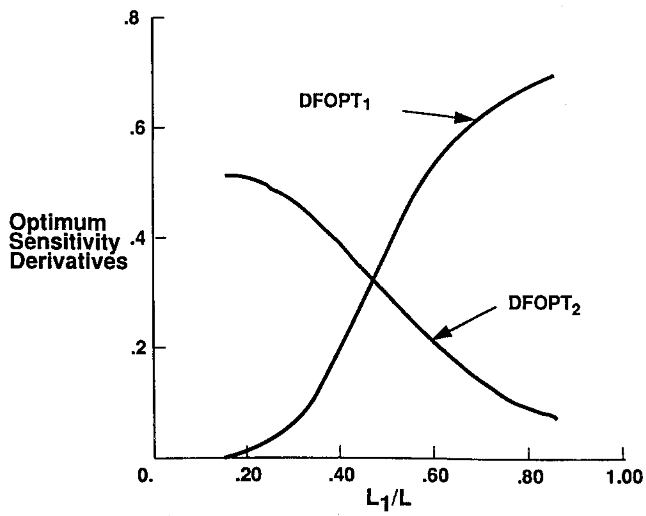
Tradeoffs Using Weighting Factors

Figure 5 shows the individual terms in the outer-loop objective function vs L_1/L. Figure 5a shows that minimizing the optimum weight drives L_1/L to 0.64. Figure 5b indicates that the OSD with respect to E_2 (DFOPT_2) drives L_1/L to the right end of the beam, but the OSD with respect to ρ_1 (DFOPT_1) drives L_1/L to the left end of the beam. Obviously, a value for L_1/L has to be chosen that represents a compromise that accounts for the conflicts.

The following study demonstrates the effect of the weighting factors on the final design. A series of discrete calculations of the outer-loop objective function OBJ vs L_1/L were carried out for two cases. In the first case (sensitivity dominated) $\beta_1 = \beta_2 = 5$, $\delta = 1$, and $W^* = 30$. In the second case (weight dominated) $\beta_1 = \beta_2 = 1$, $\delta = 5$, and $W^* = 30$. Table 3 lists the initial and final values of L_1/L, the heights of the sections, the weights of the beam, the values of OBJ, and the OSD for both cases. From Table 3 it may be seen that large values of β_1 and β_2 allow the procedure to add weight to the structure and reduce the sensitivity of the design to the uncertain parameters.



a) Weight from inner-loop optimization vs L_1/L



b) Optimum sensitivity calculated at inner-loop optimum vs L_1/L

Fig. 5 Graphs of the three components of the outer-loop objective function vs L_1/L.

Table 3 Tradeoff of insensitivity of design vs weight

$W^* = 30$ lbs		L_1/L	OBJ	h_1, in.	h_2, in.	Weight, lbs	$DFOPT_1$	$DFOPT_2$
Sensitivity dominated	Initial	0.50	4.2	0.53	3.6	51	0.38	0.32
$\delta = 1$	Final	0.43	4.1	0.40	4.0	58	0.25	0.39
$\beta_1 = \beta_2 = 5$								
Weight dominated	Initial	0.50	4.2	0.53	3.6	51	0.38	0.32
$\delta = 5$	Final	0.64	3.4	0.81	4.2	46	0.58	0.20
$\beta_1 = \beta_2 = 1$								

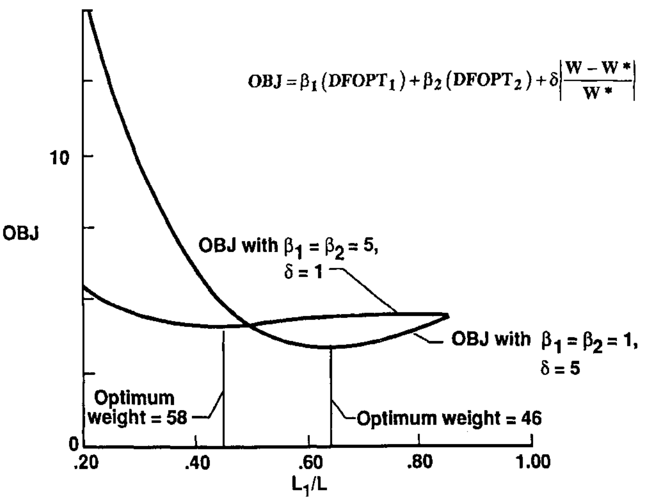


Fig. 6 Effect of weighting factors on outer-loop objective function.

Conversely, smaller values of β_1 and β_2 and a large value for δ led to the value of L_1/L that corresponds to a lower weight but much larger sensitivities. The sensitivity-dominated case obtained a 34% decrease in $DFOPT_1$, a 20% increase in $DFOPT_2$, and a 15% increase in the weight. For the weight-dominated case, a 9% decrease in the weight was achieved for a 52% increase in $DFOPT_1$ and a 38% reduction in $DFOPT_1$. Figure 6 shows graphs of OBJ vs L_1/L for both cases.

In summary, results are highly dependent on the choice of β_1, β_2, δ, and W^*; the method is yet to be tested in a large-scale application with many design variables. Nevertheless, the technique and formulation described in this paper point the way and provide the potential for a highly flexible optimization procedure that could satisfy the need for high performance combined with robustness with respect to uncertainty in parameters.

Concluding Remarks

This paper describes a method for structural optimization that minimizes the effects on a design of uncertainties in parameters normally assumed to be fixed and known. The key aspect of the procedure is that it includes sensitivity derivatives of the optimum design with respect to the uncertain parameters in the objective function. The procedure involves two nested optimization loops. The inner loop contains the primary optimization procedure (for example, minimum weight design) and the outer loop minimizes an objective function that contains the optimum value of the primary objective function and a linear combination of OSD. Each term in the outer-loop objective function is weighted by a factor to provide the flexibility to trade off inner-loop objectives vs outer-loop robustness with respect to uncertainties. The general formulation is described first, and then a simple beam example is used to illustrate the method and its potential. The example involves the design of a bimetallic beam for minimum weight with frequency constraints and seeks insensitivity to two uncertain structural properties, the density of one section and Young's modulus of the second section. Results indicate that the sensitivity of the optimum weight could be decreased by nearly a factor of 3 at the expense of a 40% increase in weight. A study was made on the effect of weighting factors. This study indicated that the designs were highly dependent on the relative values of the weighting factors. Results of the beam example also indicated that the inner-loop optima must be highly converged and OSD must be very accurately computed. The technique and formulation described in this paper provide the potential for a highly flexible optimization procedure that could satisfy the long-sought answer to the need for high performance combined with robustness with respect to the uncertainty in parameters.

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